THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING

The number of unbounded components in the Poisson-Boolean model in $\mathbb{H}^2$

Johan Tykesson

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Statistics
Chalmers University of Technology and Göteborg University
Göteborg, Sweden 2005

The number of unbounded components in the Poisson-Boolean model in $\mathbb{H}^2$
Johan Tykesson

NO 2005:30
ISSN 1652-9715

Department of Mathematical Statistics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg
Sweden
Telephone +46 (0)31 772 1000

Printed in Göteborg, Sweden 2005

The number of unbounded components in the Poisson-Boolean model in $\mathbb{H}^2$

Johan Tykesson

Department of Mathematical Statistics
Chalmers University of Technology
Göteborg University

Abstract

We consider the Poisson Boolean model with unit radius in the hyperbolic disc $\mathbb{H}^2$. Let λ be the intensity of the underlying Poisson process, and let N_C denote the number of unbounded components of the covered region. We show that there are two intensities λ_c and λ_u, $0 < \lambda_c < \lambda_u < \infty$, such that $N_C = 0$ for $\lambda \in (0, \lambda_c]$, $N_C = \infty$ for $\lambda \in (\lambda_c, \lambda_u)$, and $N_C = 1$ for $\lambda \in [\lambda_u, \infty)$. Corresponding results, due to Benjamini, Lyons, Peres and Schramm, are available for Bernoulli bond and site percolation on certain nonamenable transitive graphs, and we use many of their techniques in our proofs.

Keywords: Bernoulli percolation, Continuum percolation, Dependent percolation, Double phase transition, Hyperbolic disc, Poisson-Boolean model

Acknowledgements

First and foremost I would like to thank my advisor Associate Professor Johan Jonasson for presenting me to the topic of this Thesis, for always taking the time to help me, and for suggesting improvements on several versions of the manuscript.

I would also like to thank my assistant advisor, Professor Olle Häggström, and Professor Bo Berndtsson for their valuable remarks on earlier versions of the manuscript.

Special gratitude goes to my officemate Mattias Bengtsson for several entertaining discussions.

Finally I would also like to extend special thanks to the following colleagues: Patrik Albin, Peter Hegarty, Viktor Olsbo and Oskar Sandberg.

The number of unbounded components in the Poisson-Boolean model in $\mathbb{H}^2$

Johan Tykesson*

October 4, 2005

Abstract

We consider the Poisson-Boolean model with unit radius in the hyperbolic disc $\mathbb{H}^2$. Let λ be the intensity of the underlying Poisson process, and let N_C denote the number of unbounded components of the covered region. We show that there are two intensities λ_c and λ_u, $0 < \lambda_c < \lambda_u < \infty$, such that $N_C = 0$ for $\lambda \in (0, \lambda_c]$, $N_C = \infty$ for $\lambda \in (\lambda_c, \lambda_u)$, and $N_C = 1$ for $\lambda \in [\lambda_u, \infty)$. Corresponding results, due to Benjamini, Lyons, Peres and Schramm, are available for Bernoulli bond and site percolation on certain nonamenable transitive graphs, and we use many of their techniques in our proofs.

1 Introduction

We begin by describing the unit radius version of the so called *Poisson-Boolean* model in $\mathbb{R}^2$, arguably the most studied continuum percolation model. For a detailed study of this model, we refer to [15]. Let X be a Poisson point process in $\mathbb{R}^2$ with some intensity λ. At each point of X, place a closed ball with unit radius. Let C be the union of all balls, and V be the complement of C. The sets V and C will be referred to as the *vacant* and *covered* regions. We say that *percolation occurs* in C (respectively in V) if C (respectively V) contains unbounded (connected) components. For the Poisson-Boolean model in $\mathbb{R}^2$, it is known that there is a *critical density* $\lambda_c \in (0, \infty)$ such that for $\lambda < \lambda_c$, percolation occurs in V but not in C, and for $\lambda > \lambda_c$, percolation occurs in C but not in V. Furthermore, if we denote by N_C and N_V the number of unbounded components of C and V respectively, it is the case that $(N_C, N_V) = (0, 1)$ *a.s.* for $\lambda < \lambda_c$ and $(N_C, N_V) = (1, 0)$ *a.s.* for $\lambda > \lambda_c$. It is also known that $(N_C, N_V) = (0, 0)$ at λ_c. This means that if there is some unbounded component in C or V, it is necessarily *unique*. All these results are also valid for the Poisson-Boolean model in $\mathbb{R}^d$ for $d \geq 3$.

It is possible to consider the Poisson-Boolean in more exotic spaces than $\mathbb{R}^d$, and one might ask if there are spaces for which several unbounded components coexist with positive probability. The main result of this paper is that this is indeed the case for the hyperbolic disc, $\mathbb{H}^2$. We show that there are intensities for which there

*Research supported by the Swedish Natural Science Research Council

are almost surely infinitely many unbounded components in both the covered and vacant regions. It turns out that the main difference between $\mathbb{R}^2$ and $\mathbb{H}^2$ which causes this, is the fact that $\mathbb{H}^2$ has a *positive linear isoperimetric constant*. This means that the ratio between the circumference and area of a ball goes to some strictly positive constant as the radius goes to infinity. In $\mathbb{R}^2$ however, this ratio goes to 0.

In many aspects, the proof is similar to a proof by Benjamini, Lyons, Peres and Schramm. They showed that for a class of nonamenable planar transitive graphs, there are infinitely many infinite clusters for some parameters in Bernoulli bond percolation. We will give the proper definitions below.

The rest of the paper is organized as follows. Section 2 gives a short review of uniqueness and non-uniqueness results for infinite clusters in Bernoulli percolation on graphs, including the results by Benjamini, Lyons, Peres and Schramm. Then in Section 3 the most elementary properties of the hyperbolic disc are given. Finally, we give our proofs in Section 4.

2 Discrete percolation

Let $G = (V, E)$ be an infinite connected graph with vertex set V and edge set E. In p-Bernoulli bond percolation on G, each edge in E is kept with probability p and deleted with probability $1 - p$, independently of all other edges. All vertices are kept. Let $\mathbf{P}_p$ be the probability measure on the subgraphs of G corresponding to p-Bernoulli percolation. (It is also possible to consider p-Bernoulli site percolation in which it is the vertices that are kept or deleted, and all results we present in this section are valid in this case too.) In this section, ω will denote a random subgraph of G. Connected components of ω will be called *clusters*.

Let C be the event that p-Bernoulli bond percolation contains infinite clusters. One of the most basic facts in the theory of discrete percolation is the following theorem, a proof of which can be found in [12].

THEOREM 2.1 *There exists a critical probability $p_c = p_c(G) \in [0, 1]$ such that*

$$\mathbf{P}_p(C) = \begin{cases} 0, & p < p_c \\ 1, & p > p_c \end{cases}$$

A natural question to ask is: How many infinite clusters are there? The answer obviously depends on G and p. We will consider only *transitive* graphs.

DEFINITION 2.2 *Let $G = (V, E)$ be an infinite graph. A bijection $g : V \to V$ such that $[g(u), g(v)] \in E$ if and only if $[u, v] \in E$ is called a* graph automorphism. *The graph G is called* transitive *if for any $u, v \in V$ there exists a graph automorphism mapping u to v.*

For a transitive graph, the *degree* of the graph is the number of edges incident to each edge. The set of graph automorphisms of G is a group under composition and we denote this group by $\mathrm{Aut}(G)$. In the case of transitivity, the following theorem, see [12], gives the possible answers to the question preceding Definition 2.2.

THEOREM 2.3 *If G is transitive and ω is a p-Bernoulli bond percolation on G, the number of infinite clusters in ω is an almost sure constant which is either* 0, 1 *or* ∞.

Theorem 2.3 gives reason to introduce another quantity of interest, alongside p_c. We let $p_u = p_u(G)$ be the infimum of the set of $p \in [0,1]$ such that p-Bernoulli bond percolation has a unique infinite cluster *a.s.* Schonmann [16] showed for all transitive graphs that for all $p > p_u$, one has uniqueness. In view of Theorem 2.3 this means there are at most three phases for $p \in [0,1]$ regarding the number of infinite clusters, namely one for which this number is 0, one where the number is ∞ and finally one where uniqueness holds.

A problem which in recent years has attracted much interest is to decide for which graphs $p_c < p_u$. It turns out that whether a graph is *amenable* or not is central in settling this question:

For $K \subset V$, the *inner vertex boundary* of K is defined as $\partial_V K := \{y \in K : \exists x \notin K, [x,y] \in E\}$ and the *edge boundary* is defined as $\partial_E K := \{(x,y) : [x,y] \in E, x \in V, y \in V \setminus K\}$. The *vertex-isoperimetric* and *edge-isoperimetric* constants for G are defined as

$$\kappa_V(G) := \inf_W \frac{|\partial_V W|}{|W|} \quad \text{and} \quad \kappa_E(G) := \inf_W \frac{|\partial_E W|}{|W|}$$

where the infimum ranges over all finite connected subsets W of V.

DEFINITION 2.4 *A bounded degree graph $G = (V,E)$ is said to be* amenable *if $\kappa_V(G) = 0$ (or equivalently $\kappa_E(G) = 0$). If instead $\kappa_V(G) > 0$ we say that the graph is* nonamenable.

Benjamini and Schramm [6] have made the following general conjecture:

CONJECTURE 2.5 *If G is transitive, then $p_u > p_c$ if and only if G is nonamenable.*

Burton and Keane [7] solved one part of Conjecture 2.5:

THEOREM 2.6 *Assume that G is transitive and amenable. If ω is a p-Bernoulli percolation on G with $p > p_c$, then ω contains a unique infinite cluster a.s.*

In fact, they only proved Theorem 2.6 in the case when G is the graph with vertex set $\mathbb{Z}^2$ and whose edge set is all pairs of vertices at Euclidean distance 1 from each other, but their argument was quickly realized to work for all amenable transitive graphs. To illustrate the role of amenability, we include the main ingredients of the proof. For full details we refer to [12].

Proof. By Theorem 2.3, it is enough to rule out the case of infinitely many infinite clusters, so suppose for contradiction that the number of infinite clusters is infinite. We may assume without loss of generality that the degree of the graph is at least three. A vertex $v \in V$ is said to be a *trifurcation* if

1. v is in an infinite cluster;

2. there exist exactly three edges incident to v in ω; and

3. the deletion of x and these three edges splits the infinite cluster into exactly three disjoint infinite clusters and no finite clusters.

Let t be the probability that a given vertex is a trifurcation (by transitivity, t is independent of the choice of vertex). Since there are infinitely many infinite clusters we may choose a vertex v and $W \ni v$ so big that ∂W with positive probability contains at least three vertices belonging to three disjoint infinite clusters. Conditioned on this, there is positive probability that all edges of W are closed except three disjoint paths in ω leading from three such vertices to v. But then v is a trifurcation. So $t > 0$.

Next suppose A is a finite set of trifurcations belonging to the same infinite cluster K. A member of A is said to be an *outer* member if at least two of the disjoint infinite clusters resulting from its removal contain no other member of A. It is not difficult to show that A must contain at least one outer member. We will now show by induction on $|A|$ that the removal of all trifurcations in A will divide K into at least $|A| + 2$ disjoint infinite clusters. The claim is trivial for $|A| = 1$. Assume it holds for $|A| = j$ and suppose A is a set of $j + 1$ trifurcations in the same infinite cluster. Let v be an outer member of A. The removal of all vertices in $A \setminus \{v\}$ splits the infinite cluster into at least $j + 2$ disjoint ones. Since v is outer the removal of it gives one more infinite cluster, completing the induction.

Hence an infinite cluster with j trifurcations in a finite set $W \subset V$ must intersect ∂W in at least $j + 2$ vertices, and therefore W cannot contain more than $|\partial W| - 2$ trifurcations. Denote by $T(W)$ the number of trifurcations in W. Then, by the transitivity of G, $\mathbf{E}[T(W)] = |W|t$. Since $T(W) \leq |\partial W| - 2$, this gives

$$t \leq \frac{|\partial W| - 2}{|W|}.$$

By the amenability of G, we may choose W so that the right hand side of the above becomes arbitrarily small. Thus $t = 0$, a contradiction. □

The other direction of Conjecture 2.5 has only been partially solved. Here is one such result that will be of particular interest to us, due to Benjamini and Schramm [5]. This can be considered as the discrete analogue to our main theorem. First, another definition is needed.

DEFINITION 2.7 *Let $G = (V, E)$ be an infinite connected graph and for $W \subset V$ let N_W be the number of infinite clusters of $G \setminus W$. The number $\sup_W N_W$ where the supremum is taken over all finite W is called the* number of ends *of G.*

THEOREM 2.8 *Let G be a nonamenable, planar transitive graph with one end. Then $0 < p_c(G) < p_u(G) < 1$ for Bernoulli bond percolation on G.*

We will not discuss the condition that G can only have one end in Theorem 2.8 further. However, a homogeneous tree has infinitely many ends and $p_u = 1$.

We now review some of the results used in the proof of Theorem 2.8. In Section 4 we will prove continuous analogues to several of them.

The study of a certain kind of *dependent* percolation has produced results that have been of great help in the study of independent (Bernoulli) percolation.

Definition 2.9 *A random subgraph* ω *of* $G = (V, E)$ *is said to be an* automorphism invariant bond percolation *on* G *if* ω *has the same distribution as* $g\omega$ *for each* $g \in \mathrm{Aut}(G)$ *and the vertex set of* ω *is* V.

Clearly, usual Bernoulli bond percolation is included in this definition.

A powerful tool for handling automorphism invariant percolation is the so called *mass transport principle.*

Let $m(u, v, \omega)$ be a nonnegative function with three arguments; two vertices u and v and ω a subgraph of G. Also suppose $m(u, v, \omega) = m(gu, gv, g\omega)$ for all $u, v \in V$, all subgraphs ω and all $g \in \mathrm{Aut}(G)$. One should think of $m(u, v, \omega)$ as the amount of mass transported from u to v when the percolation results in ω. We present the mass transport principle for a certain class of transitive graphs only, namely *Cayley graphs*, since the proof in this case is simple, and will later be more easily related to the proof of the continuous analogue, Theorem 3.4.

Definition 2.10 *Let* Γ *be a finitely generated group and let* $S = \{g_1^{\pm 1}, ..., g_n^{\pm 1}\}$ *be a finite symmetric set of generators for* Γ*. The (right)* Cayley graph *of* Γ *is the graph* $G = (V, E)$ *where* $V := \Gamma$ *and* $[g, h] \in E$ *if and only if* $g^{-1}h \in S$.

Note that for each pair of elements $u, v \in \Gamma$ there is a unique element $g \in \Gamma$ such that $u = gv$. Therefore, all Cayley graphs are transitive, and Γ can be identified with a subgroup of $\mathrm{Aut}(G)$.

Theorem 2.11 *(*The mass transport principle*) If* G *is a Cayley graph,* ω *an automorphism invariant bond percolation on* G*, then for any* $u \in V$

$$\sum_{v \in V} \mathbf{E}[m(u, v, \omega)] = \sum_{v \in V} \mathbf{E}[m(v, u, \omega)].$$

Theorem 2.11 and the proof we present below is due to Benjamini, Lyons, Peres and Schramm [3]. The same authors [4] prove the mass transport principle for a wider class of transitive graphs. The first version of the mass transport principle was proved by Häggström [11], for homogeneous trees. In words, the mass transport principle says that the expected amount of mass transported out of the vertex v is the same as the expected amount of mass transported into it.

Proof. Using the automorphism invariances of ω and m and the remarks following Definition 2.10 we get

$$\sum_{u \in V} \mathbf{E}[m(u, v, \omega)] = \sum_{g \in \Gamma} \mathbf{E}[m(u, gu, \omega)] = \sum_{g \in \Gamma} \mathbf{E}[m(g^{-1}u, u, g^{-1}\omega)]$$

$$= \sum_{g \in \Gamma} \mathbf{E}[m(g^{-1}u, u, \omega)] = \sum_{u \in V} \mathbf{E}[m(v, u, \omega)],$$

completing the proof. □

Choosing the function m in different ways, the mass transport principle is used in the proofs of the following theorems from [3].

THEOREM 2.12 *Let $G = (V, E)$ be a nonamenable Cayley graph with vertex degree d and ω an automorphism invariant bond percolation on G. If $\mathbf{P}[e \in \omega] > 1 - \kappa_E(G)/d$ for all $e \in E$, then ω contains infinite clusters with positive probability.*

THEOREM 2.13 *Let G be a nonamenable Cayley graph and ω a p-Bernoulli bond percolation. If $p = p_c(G)$ there are almost surely no infinite clusters in ω.*

Later in Section 4, we will see that Theorems 2.12 and 2.13 are similar in spirit to Theorems 4.8 and 4.16 that in turn are major parts in the proof of our main theorem, Theorem 4.2.

3 The Hyperbolic Disc $\mathbb{H}^2$

The hyperbolic disc $\mathbb{H}^2$ is the open unit disc in $\mathbb{C}$ equipped with the hyperbolic metric. The hyperbolic metric is the metric which to a curve $\gamma = \{\gamma(t)\}_{t=0}^1$ assigns length

$$L(\gamma) = 2 \int_0^1 \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2} dt,$$

and to a set E assigns area

$$\mu(E) = \int_E d\sigma(z)$$

where $d\sigma(z) = 4\frac{dx\,dy}{(1-(x^2+y^2))^2}$ and $z = x + iy$. The *linear isoperimetric inequality* for $\mathbb{H}^2$ says that for all measurable $A \subset \mathbb{H}^2$ with $L(\partial A)$ and $\mu(A)$ well defined,

$$\frac{L(\partial A)}{\mu(A)} \geq 1. \tag{3.1}$$

Denote by $d(x, y)$ the hyperbolic distance between the points x and y. The closed hyperbolic ball of radius r centered at x is the set $S(x, r) := \{y : d(x, y) \leq r\}$. In what follows, area (resp. length) will always mean hyperbolic area (resp. hyperbolic length). The formulas for the area and circumference of $S(0, r)$ are given by

$$L(\partial S(0, r)) = 2\pi \sinh(r) \text{ and } \mu(S(0, r)) = 2\pi(\cosh(r) - 1). \tag{3.2}$$

Note that

$$L(\partial S(0, r)) = 2\pi r + o(r^2) \text{ as } r \to 0 \tag{3.3}$$

and

$$\mu(S(0, r)) = \pi r^2 + o(r^3) \text{ as } r \to 0. \tag{3.4}$$

Thus, at small scale, hyperbolic length and area are close to Euclidean length and area. For more elementary facts about $\mathbb{H}^2$, we refer to [8].

3.1 Mass transport

Next, we present the mass transport principle for $\mathbb{H}^2$, due to Benjamini and Schramm [5]. It is essential for our results, thus we include a proof. First some preliminary definitions are needed.

DEFINITION 3.1 *A bijective mapping of* $\mathbb{H}^2$ *onto itself that preserves (hyperbolic) distances is called an* isometry.

The set of isometries of $\mathbb{H}^2$ forms a group under composition, and we denote this group by Isom($\mathbb{H}^2$).

DEFINITION 3.2 *If A is some random subset of* $\mathbb{H}^2$*, we say that the distribution of A is* Isom($\mathbb{H}^2$)-invariant *if gA has the same distribution as A for all $g \in$*Isom($\mathbb{H}^2$).

For example, a Poisson process in $\mathbb{H}^2$ has an Isom($\mathbb{H}^2$)-invariant distribution.

DEFINITION 3.3 *A measure ν on $\mathbb{H}^2 \times \mathbb{H}^2$ is said to be* diagonally invariant *if for all measurable A, $B \subset \mathbb{H}^2$ and $g \in$*Isom($\mathbb{H}^2$)

$$\nu(gA \times gB) = \nu(A \times B).$$

THEOREM 3.4 *(*MASS TRANSPORT PRINCIPLE IN $\mathbb{H}^2$*) If ν is a positive diagonally invariant measure on $\mathbb{H}^2 \times \mathbb{H}^2$ such that $\nu(A \times \mathbb{H}^2) < \infty$ for some open $A \subset \mathbb{H}^2$, then*

$$\nu(B \times \mathbb{H}^2) = \nu(\mathbb{H}^2 \times B)$$

for all measurable $B \subset \mathbb{H}^2$.

In all our applications of this theorem, it turns out that ν is absolutely continuous with respect to $\mu \times \mu$. Below we present a proof using this assumption. It turns out that the condition $\nu(A \times \mathbb{H}^2) < \infty$ for some open A can then be dropped. For $\alpha \in \mathbb{H}^2$, let $g_\alpha(z) := (z - \alpha)/(1 - \bar{\alpha} z)$. The set of functions $\{g_\alpha\}_{\alpha \in \mathbb{H}^2}$ is a subset of Isom($\mathbb{H}^2$). Our proof requires only the diagonal invariance of ν under this subset. Note that $g_\alpha \circ g_{-\alpha}(z) = z$. The full proof of Theorem 3.4 is more involved, and we refer to [5].

The intuition behind the mass transport principle can be described as follows. One may think of $\nu(A \times B)$ as the amount of mass (or, in the case that ν is an expectation, the expected amount of mass) that goes from A to B. Thus the mass transport principle says that the amount of mass that goes out of A equals the mass that goes into A.

Proof. Suppose $\nu \ll \mu \times \mu$. By the Radon-Nikodym theorem there is f such that $\nu(A \times B) = \int_{A \times B} f \, d(\mu \times \mu)$ for all measurable $A \times B \subset \mathbb{H}^2 \times \mathbb{H}^2$. By Fubini's theorem,

$$\nu(A \times B) = \int_A \int_B f(a, b) d\mu(b) d\mu(a).$$

Also for any $g \in \text{Isom}(\mathbb{H}^2)$ we have, by $\text{Isom}(\mathbb{H}^2)$-invariance of μ,

$$\nu(gA \times gB) = \int_{gA}\int_{gB} f(a,b)d\mu(b)d\mu(a) = \int_A\int_B f(g(a),g(b))d\mu(b)d\mu(a).$$

Since ν is diagonally invariant, it follows that $f(x,y) = f(g(x),g(y))$ *a.e.* for all $g \in \text{Isom}(\mathbb{H}^2)$. Also, using $\text{Isom}(\mathbb{H}^2)$-invariance of μ again,

$$\int_{\mathbb{H}^2} f(b,x)d\mu(x) = \int_{\mathbb{H}^2} f(b,g(x))d\mu(x) \text{ and } \int_{\mathbb{H}^2} f(x,b)d\mu(x) = \int_{\mathbb{H}^2} f(g(x),b)d\mu(x)$$

for all $b \in \mathbb{H}^2$ and $g \in \text{Isom}(\mathbb{H}^2)$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{H}^2} f(b,x)d\mu(x) = \int_{\mathbb{H}^2} f(0,g_b(x))d\mu(x) = \int_{\mathbb{H}^2} f(0,x)d\mu(x) \\
&= \int_{\mathbb{H}^2} f(g_x(0),g_x(x))d\mu(x) = \int_{\mathbb{H}^2} f(-x,0)d\mu(x) = \int_{\mathbb{H}^2} f(x,0)d\mu(x) \\
&= \int_{\mathbb{H}^2} f(g_{-b}(x),g_{-b}(0))d\mu(x) = \int_{\mathbb{H}^2} f(g_{-b}(x),b)d\mu(x) = \int_{\mathbb{H}^2} f(x,b)d\mu(x).
\end{aligned}$$

Using Fubini again we get

$$\nu(B \times \mathbb{H}^2) = \int_B\int_{\mathbb{H}^2} f(b,x)d\mu(x)d\mu(b) = \int_B\int_{\mathbb{H}^2} f(x,b)d\mu(x)d\mu(b) = \nu(\mathbb{H}^2 \times B)$$

completing the proof.□

4 The Poisson-Boolean model in $\mathbb{H}^2$

Definition 4.1 *A point process X on $\mathbb{H}^2$ distributed according to the probability measure $\mathbf{P}$ such that for $k \in \mathbb{N}$, $\lambda \geq 0$, and every measurable $A \subset \mathbb{H}^2$ one has*

$$\mathbf{P}[|X(A)| = k] = e^{-\lambda\mu(A)}\frac{(\lambda\mu(A))^k}{k!}$$

is called a Poisson process *with intensity λ on $\mathbb{H}^2$. Here $X(A) = X \cap A$ and $|\cdot|$ denotes cardinality.*

In the *Poisson-Boolean model* in $\mathbb{H}^2$, just like in the corresponding model in $\mathbb{R}^2$, at every point of a Poisson process X we place a ball with unit radius. More precisely, we let $C = \bigcup_{x\in X} S(x,1)$ and $V = C^c$ and refer to C and V as the covered and vacant regions of $\mathbb{H}^2$ respectively. For $A \subset \mathbb{H}^2$ we let $C[A] := \bigcup_{x\in X(A)} S(x,1)$ and $V[A] := C[A]^c$. For $x, y \in \mathbb{H}^2$, let $d_C(x,y)$ be the length of the shortest curve connecting x and y lying completely in C if there exists such a curve, otherwise let $d_C(x,y) = \infty$. Similarly, let $d_V(x,y)$ be the length of the shortest curve connecting x and y lying completely in V if there is such a curve, otherwise let $d_V(x,y) = \infty$. The collection of all components of C is denoted by $\mathcal{C}$ and the collection of all components of V is denoted by $\mathcal{V}$. Let N_C denote the number of unbounded components in C

and N_V denote the number of unbounded components in V. Next we introduce *critical densities* as follows. We let

$$\lambda_c := \inf\{\lambda \,:\, N_C > 0 \text{ a.s.}\},$$

$$\lambda_u = \inf\{\lambda \,:\, N_C = 1 \text{ a.s.}\},$$

$$\lambda_c^* = \sup\{\lambda \,:\, N_V > 0 \text{ a.s. }\},$$

and

$$\lambda_u^* = \sup\{\lambda \,:\, N_V = 1 \text{ a.s. }\}.$$

Our main result is:

THEOREM 4.2 *For the Poisson-Boolean model with unit radius in* $\mathbb{H}^2$

$$0 < \lambda_c < \lambda_u < \infty.$$

Furthermore, with probability 1,

$$(N_C, N_V) = \begin{cases} (0,1), & \lambda \in [0, \lambda_c] \\ (\infty, \infty), & \lambda \in (\lambda_c, \lambda_u) \\ (1,0), & \lambda \in [\lambda_u, \infty) \end{cases}$$

A first step towards Theorem 4.2 is given by the below lemma.

LEMMA 4.3 *For the Poisson-Boolean model in* $\mathbb{H}^2$*,* $\lambda_c^* < \infty$ *and* $\lambda_c > 0$.

Proof. Let Γ be a regular tiling of $\mathbb{H}^2$ into congruent polygons of finite diameter. The polygons of Γ can be identified with the vertices of a planar nonamenable transitive graph $G = (V, E)$. Next, we define a Bernoulli site percolation ω on G. We declare each vertex $v \in V$ to be in ω if and only if its corresponding polygon $\Gamma(v)$ is not completely covered by $C[\Gamma(v)]$. Clearly, the vertices are declared to be in ω or not with the same probability and independently of each other. Now for any v,

$$\lim_{\lambda \to \infty} \mathbf{P}[v \text{ is in } \omega] = 0.$$

Thus, by Theorem 2.8, for λ large enough, there are no infinite clusters in ω. But if there are no infinite clusters in ω, there are no unbounded components of V. Thus $\lambda_c^* < \infty$.

To show $\lambda_c > 0$ we adapt an argument due to Hall [13]. Construct a branching process, whose members are points in $\mathbb{H}^2$, as follows. The individual in the 0'th generation is taken to be the center of a ball with unit radius. Without loss of generality the center can be taken to be the origin. Given individuals $Z_{n1}, Z_{n2}, ..., Z_{nN_n}$ in the n:th generation, the $(n+1)$:th generation is defined as follows. For $l = 1, ..., N_n$ let X_{nl} be a Poisson process with intensity λ, independent of the previous history of the branching process and also of $X_{nl'}$ for $l \neq l'$. At each point of X_{nl} center a ball of unit radius. The progeny of Z_{nl} is then taken to be the points of X_{nl} whose associated balls intersect that of Z_{nl}. The number of descendants of Z_{nl} clearly has a Poisson distribution with expectation $\lambda\mu(S(0,2))$. Therefore, the expected number of individuals in generation n is given by $\lambda^n\mu(S(0,2))^n$ and consequently, the expected number of individuals in the whole branching process equals $\sum_{n=1}^{\infty}\lambda^n\mu(S(0,2))^n$. Thus if $\lambda < \mu(S(0,2))^{-1} \approx 0.0567$, the expected total number of individuals is finite. However, the expected number of individuals in the branching process is greater than or equal to the expected number of balls in a component of the covered region in the Poisson-Boolean model. Thus $\lambda_c > 0.056$. □

4.1 FKG inequality

As in the theory for discrete percolation, a correlation inequality for *increasing* and *decreasing* events turns out to be very useful. If ω and ω' are two realizations of a Poisson-Boolean model we write $\omega \preceq \omega'$ if any ball present in ω is also present in ω'.

Definition 4.4 *An event A is said to be* increasing *(respectively* decreasing*) if $\omega \preceq \omega'$ implies $1_A(\omega) \le 1_A(\omega')$ (respectively $1_A(\omega) \ge 1_A(\omega')$).*

Here we present the FKG inequality for the fixed radius version of the Poisson-Boolean model in $\mathbb{H}^2$. The proof is very similar to the proof of the corresponding theorem in $\mathbb{R}^2$, Theorem 2.2 from [15], but requires a minor modification.

Theorem 4.5 *(*FKG inequality*) If A and B are both increasing or both decreasing events, then $\mathbf{P}[A \cap B] \ge \mathbf{P}[A]\mathbf{P}[B]$.*

Proof. Let $\{G_n\}_{n=1}^{\infty}$ be a sequence of tilings of $\mathbb{H}^2$ into cells of equal area such that G_n is obtained by splitting the cells of G_{n-1} into smaller cells, and

$$\lim_{n\to\infty} (\sup\{\mathrm{diam}(\Xi) : \Xi \text{ is a cell in } G_n\}) = 0.$$

We may take G_1 to be the same as in the proof of Lemma 4.3. For each cell Ξ in G_n, let $N_n(\Xi) = 1$ if there is a Poisson point in Ξ and 0 otherwise. Let $\mathcal{F}_n$ be the σ-algebra generated by the random variables $\{N_n(\Xi) : \Xi \text{ is a cell in } G_n\}$. Then, for any event A which is defined in terms of the Poisson process, $\{\mathbf{E}[\mathbf{1}_A|\mathcal{F}_n]\}_{n=1}^{\infty}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n=1}^{\infty}$. Set $\mathcal{F}_\infty := \sigma(\cup_{n=1}^{\infty}\mathcal{F}_n)$. Clearly A is measurable with respect to $\mathcal{F}_\infty$. Now Lévy's upwards theorem gives

$$\lim_{n\to\infty} \mathbf{E}[\mathbf{1}_A|\mathcal{F}_n] = \mathbf{E}[\mathbf{1}_A|\mathcal{F}_\infty] = \mathbf{1}_A \text{ a.s.} \tag{4.1}$$

It is clear that for any n, any $\omega \preceq \omega'$, and any increasing event A, $\mathbf{E}[\mathbf{1}_A|\mathcal{F}_n](\omega) \le \mathbf{E}[\mathbf{1}_A|\mathcal{F}_n](\omega')$. Also it is obvious that the random variables $N_n(C)$ are all independent. Therefore, for any two increasing events A_1 and A_2, the usual (discrete) FKG inequality (see Theorem 2.4 in [10]) gives

$$\begin{aligned}\mathbf{E}[\mathbf{E}[\mathbf{1}_{A_1}|\mathcal{F}_n]\mathbf{E}[\mathbf{1}_{A_2}|\mathcal{F}_n]] &\ge \mathbf{E}[\mathbf{E}[\mathbf{1}_{A_1}|\mathcal{F}_n]]\mathbf{E}[\mathbf{E}[\mathbf{1}_{A_2}|\mathcal{F}_n]] \\ &= \mathbf{E}[\mathbf{1}_{A_1}]\mathbf{E}[\mathbf{1}_{A_2}].\end{aligned}$$

The dominated convergence theorem and (4.1) give

$$\lim_{n\to\infty} \mathbf{E}[\mathbf{E}[\mathbf{1}_{A_1}|\mathcal{F}_n]\mathbf{E}[\mathbf{1}_{A_2}|\mathcal{F}_n]] = \mathbf{E}[\mathbf{1}_{A_1}\mathbf{1}_{A_2}],$$

completing the proof. □

We will also use the following simple corollary to Theorem 4.5, the proof of which can be found in [10].

COROLLARY 4.6 *(*THE SQUARE ROOT TRICK*) If A_1, A_2, ..., A_m are increasing events with the same probability, then*

$$\mathbf{P}[A_1] \geq 1 - \left(1 - \mathbf{P}[\cup_{i=1}^m A_i]\right)^{1/m}.$$

The same holds when A_1, A_2, ..., A_m are decreasing.

4.2 The number of unbounded components

The aim of this section is to determine the possible values of (N_C, N_V). The first lemma is an application of the mass transport principle. First, some notation is needed. We write $H \in \mathfrak{H}$ if H is a union of elements from $\mathcal{C}$ and $\mathcal{V}$ such that its distribution is Isom($\mathbb{H}^2$)-invariant, and let $\mathcal{H}$ denote the collection of all components of H. For $h \in \mathcal{H}$ and sets A, $B \subset \mathbb{H}^2$ we write $A \overset{h}{\leftrightsquigarrow} B$ if h intersects both A and B.

LEMMA 4.7 *If $H \in \mathfrak{H}$ contains only finite components a.s., then for any measurable A*

$$\mathbf{E}[\mu(A \cap H)] \leq \mathbf{E}[L(A \cap \partial H)].$$

Before the proof we describe the intuition behind it: We place mass of unit density in all of $\mathbb{H}^2$. Then, if h is a component of H, the mass inside h is transported to the boundary of h. Then we use the mass transport principle: the expected amount of mass transported out of a subset A equals the expected amount of mass transported into it. Finally we combine this with the isoperimetric inequality (3.1). *Proof.* For A, $B \subset \mathbb{H}^2$ and $H \in \mathfrak{H}$, let

$$\eta(A \times B,\, H) := \sum_{h \in \mathcal{H}\,:\, A \overset{h}{\leftrightsquigarrow} B} \frac{\mu(B \cap h)\, L(A \cap \partial h)}{L(\partial h)}.$$

and let $\nu(A \times B) := \mathbf{E}[\eta(A \times B,\, H)]$. Since the distribution of H is Isom($\mathbb{H}^2$)-invariant, we get for each $g \in$Isom($\mathbb{H}^2$)

$$\nu(gA \times gB) = \mathbf{E}[\eta(gA \times gB,\, H)] = \mathbf{E}[\eta(gA \times gB,\, gH)]$$

$$= \mathbf{E}[\eta(A \times B,\, H)] = \nu(A \times B).$$

Thus, ν is a diagonally invariant positive measure on $\mathbb{H}^2 \times \mathbb{H}^2$. We have $\nu(\mathbb{H}^2 \times A) = \mathbf{E}\left[\mu(A \cap H)\right]$ and

$$\nu(A \times \mathbb{H}^2) = \mathbf{E}\left[\sum_{h \in \mathcal{H}\,:\, A \overset{h}{\leftrightsquigarrow} \mathbb{H}^2} \frac{\mu(h)\, L(A \cap \partial h)}{L(\partial h)}\right] \leq \mathbf{E}[L(A \cap \partial H)]$$

where the last inequality follows from the linear isoperimetric inequality. Hence, the claim follows by Theorem 3.4. $\square$

We remark that obviously Lemma 4.7 holds for many other objects, that have a distribution which is $\mathrm{Isom}(\mathbb{H}^2)$ invariant.

LEMMA 4.8 *Suppose $H \in \mathfrak{H}$. If $\mathbf{P}[0 \in H] > (1 + 1/e)/(-1 + e) \approx 0.796$ then H contains unbounded components with positive probability.*

Note that $\mathbf{P}[z \in H]$ is the same for all $z \in \mathbb{H}^2$ since $H \in \mathfrak{H}$. *Proof.* We assume that H contains only finite components almost surely and use Lemma 4.7. Obviously we have

$$\lim_{\epsilon \to 0} \mathbf{P}[S(0, \epsilon) \cap \partial H \neq \emptyset] = 0$$

and

$$\lim_{\epsilon \to 0} \mathbf{P}[S(0, \epsilon) \subset H] = \mathbf{P}[0 \in H].$$

Therefore, using (3.4), we conclude

$$\mathbf{E}[\mu(S(0, \epsilon) \cap H)] = \mathbf{P}[0 \in H]\pi\epsilon^2 + o(\epsilon^2) \text{ as } \epsilon \to 0. \tag{4.2}$$

In the same way, for small $\epsilon > 0$, the probability that $S(0, \epsilon)$ intersects more than one component of H is small compared to the probability that $S(0, \epsilon)$ intersects one component. Also when ϵ is small, conditioned on the event $\{S(0, \epsilon) \cap \partial H \neq \emptyset\}$, $S(0, \epsilon) \cap \partial H$ will be close to a straight line such that the distance from its middle point to the origin is uniformly distributed between 0 and ϵ. Thus

$$\mathbf{E}[L(S(0, \epsilon) \cap \partial H) |\, S(0, \epsilon) \cap \partial H \neq \emptyset] =$$

$$\int_0^\epsilon 2\frac{\sqrt{\epsilon^2 - x^2}}{\epsilon}\, dx + o(\epsilon) = \frac{\pi}{2}\epsilon + o(\epsilon) \text{ as } \epsilon \to 0. \tag{4.3}$$

Using the independence of the Poisson process, the obvious fact that $\partial H \subset \partial C$ and formulas (3.2) we get

$$\begin{aligned}
&\mathbf{P}\left[S(0, \epsilon) \cap \partial H \neq \emptyset\right] \leq \mathbf{P}\left[S(0, \epsilon) \cap \partial C \neq \emptyset\right] \\
&\leq \mathbf{P}[\{|X(S(0, 1 + \epsilon) \backslash S(0, 1 - \epsilon))| > 0\} \cap \{|X(S(0, 1 - \epsilon))| = 0\}] \\
&= \mathbf{P}[|X(S(0, 1 + \epsilon) \backslash S(0, 1 - \epsilon))| > 0]\, \mathbf{P}[|X(S(0, 1 - \epsilon))| = 0] \\
&= (1 - \exp\left(-\lambda\, \mu(S(0, 1 + \epsilon) \backslash S(0, 1 - \epsilon))\right) \exp\left(-\lambda\, \mu(S(0, 1 - \epsilon))\right)
\end{aligned}$$

$$= \exp\left(-2\pi\lambda(\cosh(1-\epsilon)-1)\right) - \exp\left(-2\pi\lambda(\cosh(1+\epsilon)-1)\right)$$

$$= 4\exp\left(2\lambda\pi - 2\lambda\pi\cosh(1)\right)\lambda\pi\sinh(1)\epsilon + o(\epsilon^2) \text{ as } \epsilon \to 0. \tag{4.4}$$

Hence, by (4.3) and (4.4),

$$\mathbf{E}\left[L(S(0,\epsilon)\cap\partial H)\right] \le 2\exp\left(2\lambda\pi - 2\lambda\pi\cosh(1)\right)\lambda\pi^2\sinh(1)\epsilon^2 + o(\epsilon^2) \text{ as } \epsilon \to 0. \tag{4.5}$$

By Lemma 4.7, $\mathbf{E}[\mu(S(0,\epsilon)\cap H)] \le \mathbf{E}[L(S(0,\epsilon)\cap\partial H)]$, so by (4.2) and (4.5) it follows that

$$\mathbf{P}[0 \in H] \le 2\exp\left(2\lambda\pi - 2\lambda\pi\cosh(1)\right)\lambda\pi\sinh(1). \tag{4.6}$$

By straightforward calculations, the right hand side in (4.6) is at most $(1+1/e)/(-1+e)$ for all λ. This completes the proof. □

Lemma 4.9 *N_C is an almost sure constant which equals 0, 1 or ∞.*

Proof. First we show, following [12], that N_C is an *a.s.* constant. For $n \in \{0,1,2,...\} \cup \{\infty\}$ let D_n be the event that $N_C = n$. Assume for contradiction that there is n such that

$$0 < \mathbf{P}[D_n] < 1 \tag{4.7}$$

and fix such an n. For a point z in $\mathbb{H}^2$ and a positive integer k, let

$$\mathbf{1}_{n,z,k} := \begin{cases} 0 & \text{if } \mathbf{P}[D_n|X(S(z,k))] \le 1/2 \\ 1 & \text{if } \mathbf{P}[D_n|X(S(z,k))] > 1/2 \end{cases}$$

Thus $\mathbf{1}_{n,z,k}$ is the best guess of $\mathbf{1}_{D_n}$ given the configuration of the Poisson process in $S(z,k)$. By Lévy's 0-1-law (see [9], page 263) we get for fixed z that

$$\lim_{k\to\infty} \mathbf{1}_{n,z,k} = \mathbf{1}_{D_n} \text{ a.s.} \tag{4.8}$$

Let $z_1, z_2, ...$ be a sequence of points such that for each k, $S(z,k)$ and $S(z_k,k)$ do not intersect. Since $(\mathbf{1}_{D_n}, \mathbf{1}_{n,z_k,k})$ has the same joint distribution as $(\mathbf{1}_{D_n}, \mathbf{1}_{n,z,k})$, we get from (4.8) that $\mathbf{1}_{n,z_k,k}$ converges in probability to $\mathbf{1}_{D_n}$ as $k \to \infty$. Thus

$$\lim_{k\to\infty} \mathbf{P}[\mathbf{1}_{n,z_k,k} = \mathbf{1}_{n,z,k} = \mathbf{1}_{D_n}] = 1. \tag{4.9}$$

But since $S(z,k)$ and $S(z_k,k)$ are disjoint, $\mathbf{1}_{n,z,k}$ and $\mathbf{1}_{n,z_k,k}$ are independent random variables. Thus, using (4.7), we get

$$\lim_{k\to\infty} \mathbf{P}[\mathbf{1}_{n,z_k,k} = 1 = 1 - \mathbf{1}_{n,z,k}] = \lim_{k\to\infty} \mathbf{P}[\mathbf{1}_{n,z_k,k} = 1]\mathbf{P}[\mathbf{1}_{n,z,k} = 0]$$

$$= \mathbf{P}[D_n](1 - \mathbf{P}[D_n]) > 0.$$

This contradicts (4.9), thus the assumption (4.7) is false, and there is n such that $\mathbf{P}[D_n] = 1$. Next, we show that this n must be in $\{0, 1, \infty\}$. Suppose $2 \le n < \infty$. Since n is finite, it is possible to pick a large $R > 0$ such that the event

$\{S(x,R)$ intersects every unbounded component of $C\}$ has positive probability for $x \in \mathbb{H}^2$. With this R, we can then pick $\epsilon > 0$ small such that the event

$$A := \{S(x, R+1-\epsilon) \text{ intersects every unbounded component } U \text{ of } C[S(x,R)^c]\}$$

has positive probability. Let $E := \{S(x, R+1-\epsilon) \subset C[S(x,R)]\}$. Clearly $\mathbf{P}[E] > 0$. Since A depends only on $X(S(x,R)^c)$ and E depends only on $X(S(x,R))$, they are independent. Hence, $\mathbf{P}[A \cap E] > 0$. But on $A \cap E$, there is only one unbounded component of C, a contradiction. Therefore, $n \in \{0, 1, \infty\}$. □

Corollary 4.10 *For the Poisson-Boolean model in* $\mathbb{H}^2$, $\lambda_c < 0.407$.

Proof. Since $\mathbf{P}[0 \in C] = \mathbf{P}[|X(S(0,1))| > 0] = 1 - \exp(-2\pi\lambda(\cosh(1) - 1))$

$$> 2\exp(2\lambda\pi - 2\lambda\pi\cosh(1))\lambda\pi\sinh(1)$$

if $\lambda > 0.4063$, the desired conclusion follows from (4.6) and Lemma 4.9.□

The next Lemma is proved in the same fashion as Lemma 4.9.

Lemma 4.11 N_V *is an almost sure constant which equals* 0, 1 *or* ∞.

Proof. If D_n is the event that $N_V = n$, it follows in the same way as in the proof of Lemma 4.9 that there is n such that $\mathbf{P}[D_n] = 1$, and it remains to show that this $n \in \{0, 1, \infty\}$. Suppose $2 \le n < \infty$ is an integer and $N_V = n$ a.s. Pick $R > 0$ such that the event

$$A := \{S(0,R) \text{ intersects all unbounded components } U \text{ of } V\}$$

has positive probability, which is possible since n is finite. Removing finitely many points from X and associated balls does not increase the number of unbounded vacant components. Thus

$$B := \{S(0,R) \text{ intersects all unbounded components } U \text{ of } V[S(0,R+1)^c]\}.$$

also has positive probability. Let $D = \{|X(S(0,R+1))| = 0\}$. Since B and D are independent and D has positive probability, $B \cap D$ has positive probability. But on $B \cap D$ there is only one unbounded component of V. This contradicts the initial assumption, completing the proof. □

Lemma 4.12 *For* $H \in \mathfrak{H}$, H *and/or* H^c *contains unbounded components almost surely.*

Proof. Suppose H and $D := H^c$ contains only finite components, and let in this proof $\mathcal{H}_0$ and $\mathcal{D}_0$ be the collections of the components of H and D respectively. Then every element h of $\mathcal{H}_0$ is surrounded by a unique element h' of $\mathcal{D}_0$, which in turn is surrounded by a unique element h'' of $\mathcal{H}_0$. In the same way, every element d of $\mathcal{D}_0$ is surrounded by a unique element d' of $\mathcal{H}_0$ which in turn is surrounded by a unique element d'' of $\mathcal{D}_0$. Inductively, for $j \in \mathbb{N}$, let $\mathcal{H}_{j+1} := \{h'' : h \in \mathcal{H}_j\}$ and $\mathcal{D}_{j+1} := \{d'' : d \in \mathcal{D}_j\}$. Next, for $r \in \mathbb{N}$, let

$$A_r := \bigcup_{j=0}^{r} (\{h \in \mathcal{H}_0 : \sup\{i : h \in \mathcal{H}_i\} = j\} \cup \{d \in \mathcal{D}_0 : \sup\{i : d \in \mathcal{D}_i\} = j\}).$$

In words, $\mathcal{H}_j$ and $\mathcal{D}_j$ define layers of components from H and D. Thus A_r is the union of all layers of components from H and D that have at most r layers inside of them. Obviously $A_r \in \mathfrak{H}$ for all r and

$$\lim_{r\to\infty} \mathbf{P}[0 \in A_r] = 1.$$

Hence, by (4.8), there is R such that for $r \geq R$,

$$\mathbf{P}[A_r \text{ has unbounded components}] > 0.$$

But by construction, for any r, A_r has only finite components. Hence the initial assumption is false.□

LEMMA 4.13 *The cases $(N_C, N_V) = (\infty, 1)$ and $(N_C, N_V) = (1, \infty)$ have probability* 0.

Proof. Suppose $N_C = \infty$. First we show that it is possible to pick $R > 0$ such that the event

$$A(x, R) :=$$

$$\{S(x, R) \text{ intersects at least 2 disjoint unbounded components of } C[S(x, R)^c]\}$$

has positive probability for $x \in \mathbb{H}^2$. Suppose $S(x, r)$ intersects an unbounded component of C for some $r > 0$. Then if $S(x, r)$ does not intersect some unbounded component of $C[S(x, r)^c]$, there must be some ball centered in $S(x, r+2)\setminus S(x, r+1)$ being part of an unbounded component of $C[S(x, r+1)^c]$, which is to say that $S(x, r+1)$ intersects an unbounded component of $C[S(x, r+1)^c]$. Clearly can find $\tilde{R}$ such that

$$B(x, \tilde{R}) :=$$

$$\{S(x, \tilde{R}) \text{ intersects at least 3 disjoint unbounded components of } C\}.$$

By the above discussion it follows that $\mathbf{P}[A(x, \tilde{R}) \cup A(x, \tilde{R}+1)] > 0$, which proves the existence of R such that $A(x, R)$ has positive probability. Pick such an R and let $E(x, R) := \{S(x, R) \subset C[S(x, R)]\}$. E has positive probability and is independent of A so $A \cap E$ has positive probability. By planarity, on $A \cap E$, V contains at least 2 unbounded components. So with positive probability, $N_V > 1$. By Lemma 4.11, $N_V = \infty$ a.s. This finishes the first part of the proof.

Now instead suppose $N_V = \infty$ and pick $R > 0$ such that

$$A(x, R) := \{S(x, R) \text{ intersects at least two unbounded components } U \text{ of } V\}$$

has positive probability. Let

$$B(x,R) := \{C[S(x,R+1)^c] \text{ contains at least 2 unbounded components}\}.$$

On A, $C\backslash S(0,R)$ contains at least two unbounded components, which in turn implies that B occurs. Since $\mathbf{P}[A] > 0$ this gives $\mathbf{P}[B] > 0$. Since B is independent of $F(x,R) := \{|X(S(x,R+1))| = 0\}$ which has positive probability, $\mathbf{P}[B \cap F] > 0$. On $B \cap F$, C contains at least two unbounded components. By Lemma 4.9 we get $N_C = \infty$ a.s. $\square$

LEMMA 4.14 *The case* $(N_C, N_V) = (1,1)$ *has probability* 0.

Proof. Assume $(N_C, N_V) = (1,1)$ a.s. Fix $x \in \mathbb{H}^2$. Denote by $A^u_C(R)$ (respectively $A^d_C(R)$, $A^r_C(R)$, $A^l_C(R)$) the event that the uppermost (respectively lowermost, rightmost, leftmost) quarter of $\partial S(x,R)$ intersects an unbounded component of $C\backslash S(x,R)$. Clearly, these events are increasing. Since $N_C = 1$ a.s.,

$$\lim_{R\to\infty} \mathbf{P}[A^u_C(R) \cup A^d_C(R) \cup A^r_C(R) \cup A^l_C(R)] = 1.$$

Hence by Corollary 4.6, $\lim_{R\to\infty} \mathbf{P}[A^t_C(R)] = 1$ for $t \in \{u,\, d,\, r,\, l\}$. Now let $A^u_V(R)$ (respectively $A^l_V(R)$, $A^r_V(R)$, $A^l_V(R)$) be the event that the uppermost (respectively lowermost, rightmost, leftmost) quarter of $\partial S(x,R)$ intersects an unbounded component of $V\backslash S(x,R)$. Since these events are decreasing, we get in the same way as above that $\lim_{R\to\infty} \mathbf{P}[A^t_V(R)] = 1$ for $t \in \{u,\, d,\, r,\, l\}$. Thus we may pick R_1 so big that $\mathbf{P}[A^t_C(R_1)] > 7/8$ and $\mathbf{P}[A^t_V(R_1)] > 7/8$ for $t \in \{u,\, d,\, r,\, l\}$. Let

$$A := A^u_C(R_1) \cap A^d_C(R_1) \cap A^l_V(R_1) \cap A^r_V(R_1).$$

Bonferroni's inequality implies $\mathbf{P}[A] > 1/2$. On A, $C\backslash S(x,R)$ contains two disjoint unbounded components. Since $N_C = 1$ a.s., these two components must almost surely on A be connected. The existence of such a connection implies that there are at least two unbounded components of V, an event with probability 0. This gives $\mathbf{P}[A] = 0$, a contradiction. $\square$

PROPOSITION 4.15 *Almost surely,* $(N_C, N_V) \in \{(1,0),\, (0,1),\, (\infty,\infty)\}$.

Proof. By Lemmas 4.9 and 4.11, each of N_C and N_V is in $\{0,\, 1,\, \infty\}$. Lemma 4.12 with $H \equiv C$ rules out the case $(0,0)$. Hence Lemmas 4.13 and 4.14 imply that it remains only to rule out the cases $(0,\infty)$ and $(\infty,0)$. But since every two unbounded components of C must be separated by some unbounded component of V, $(\infty,0)$ is impossible. In the same way, $(0,\infty)$ is impossible. $\square$

4.3 The situation at λ_c and λ_c^*

It turns out that to prove the main theorem, it is necessary to investigate what happens regarding N_C and N_V at the intensities λ_c and λ_c^*. Our proofs are inspired by the proof of Theorem 1.1 in [3], which says that critical Bernoulli percolation on nonamenable Cayley graphs does not contain infinite clusters. Notably, for the Poisson Boolean model in $\mathbb{R}^2$, it is the case that $(N_C, N_V) = (0,0)$ a.s. at λ_c (see [1]). By Proposition 4.15, this is not possible in $\mathbb{H}^2$.

THEOREM 4.16 *At λ_c, $N_C = 0$ a.s.*

Proof. We begin with ruling out the possibility of a unique unbounded component of C at λ_c. Suppose $\lambda = \lambda_c$ and that $N_C = 1$ a.s. Denote the unique unbounded component of C by U. By Proposition 4.15, V contains only finite components a.s. Let $\epsilon > 0$ be small and remove each point in X with probability ϵ and denote by X_ϵ the remaining points. Furthermore, let $C_\epsilon = \cup_{x \in X_\epsilon} S(x, 1)$. Since X_ϵ is a Poisson process with intensity $\lambda_c - \epsilon$ it follows that C_ϵ will contain only bounded components a.s. Let $\mathcal{C}_\epsilon$ be the collection of all components of C_ϵ. We will now construct H_ϵ as a union of elements from $\mathcal{C}_\epsilon$ and $\mathcal{V}$ such that the distribution of H_ϵ will be Isom($\mathbb{H}^2$)-invariant. For each $z \in \mathbb{H}^2$ we let $U_\epsilon(z)$ be the union of the components of $U \cap C_\epsilon$ being closest to z. We let each h from $\mathcal{C}_\epsilon \cup \mathcal{V}$ be in H_ϵ if and only if $\sup_{z \in h} d(z, U) < 1/\epsilon$ and $U_\epsilon(x) = U_\epsilon(y)$ for all x, $y \in h$. It is now clear for almost every realisation of the underlying Poisson process X,

$$\lim_{\epsilon \to 0} \mathbf{P}[0 \in H_\epsilon | X] = 1.$$

Hence the Bounded Convergence Theorem gives

$$\lim_{\epsilon \to 0} \mathbf{P}[0 \in H_\epsilon] = 1.$$

Since H_ϵ is not a union of elements from $\mathcal{C}$ and $\mathcal{V}$, Lemma 4.8 is not directly applicable. However, as $\epsilon \to 0$, $(\partial C \cup \partial C_\epsilon) \downarrow \partial C$. By inspecting the proof of Lemma 4.8 (the calculation leading to (4.4)), we see that this is enough to conclude that if $\mathbf{P}[0 \in H_\epsilon]$ is close enough to 1, H_ϵ contains unbounded components with positive probability. Suppose $h_1, h_2, \ldots$ is an infinite sequence of distinct elements from $\mathcal{C}_\epsilon \cup \mathcal{V}$ such that they constitute an unbounded component of H_ϵ. Then $U_\epsilon(x) = U_\epsilon(y)$ for all x, y in this component. Hence $U \cap C_\epsilon$ contains an unbounded component (this particular conclusion could not have been made without the condition $\sup_{z \in h} d(z, U) < 1/\epsilon$ in the definition of $U_\epsilon(z)$). Therefore we conclude that the existence of an unbounded component in H_ϵ implies the existence of an unbounded component in C_ϵ. Hence C_ϵ contains an unbounded component with positive probability, a contradiction.

We move on to rule out the case of infinitely many unbounded components of C at λ_c. Assume $N_C = \infty$ a.s. at λ_c. As in the proof of Lemma 4.13, we choose R such that for $x \in \mathbb{H}^2$ the event

$$A(x, R) :=$$

$$\{S(x, R) \text{ intersects at least 3 disjoint unbounded components of } C[S(x, R)^c]\}$$

has positive probability. Let $B(x, R) := \{S(x, R) \subset C[S(x, R)]\}$ for $x \in \mathbb{H}^2$. Since A and B are independent, it follows that $A \cap B$ has positive probability. On $A \cap B$, x is contained in an unbounded component U of C. Furthermore, $U \backslash S(x, R + 1)$ contains at least three disjoint unbounded components. Now let Y be a Poisson process independent of X with some positive intensity. We call a point $y \in \mathbb{H}^2$ a *encounter point* if

- $y \in Y$;
- $A(y, R) \cap B(y, R)$ occurs;

- $S(y, 2(R+1)) \cap Y = \{y\}$.

The third condition above means that if y_1 and y_2 are two encounter points, then $S(y_1, R+1)$ and $S(y_2, R+1)$ are disjoint sets. By the above, it is clear that given $y \in Y$, the probability that y is an encounter point is positive. We now move on to show that if y is an encounter point and U is the unbounded component of C containing y, then each of the disjoint unbounded components of $U \backslash S(y, R+1)$ contains a further encounter point.

Let $m(s,t) = 1$ if t is the unique encounter point closest to s in C, and $m(s,t) = 0$ otherwise. Then let for measurable sets A, $B \subset \mathbb{H}^2$

$$\eta(A \times B,\, X,\, Y) = \sum_{s \in Y(A)} \sum_{t \in Y(B)} m(s,t)$$

and

$$\nu(A \times B) = \mathbf{E}[\eta(A \times B,\, X,\, Y)].$$

Clearly, ν is a positive diagonally invariant measure on $\mathbb{H}^2 \times \mathbb{H}^2$. Suppose A is some ball in $\mathbb{H}^2$. Since $\sum_{t \in Y} m(s,t) \leq 1$ we get
$\nu(A \times \mathbb{H}^2) \leq \mathbf{E}[|Y(A)|] < \infty$. On the other hand, if y is an encounter point lying in A and with positive probability there is no encounter point in some of the unbounded components of $U \backslash S(y, R+1)$ we get $\sum_{s \in Y} \sum_{t \in Y(A)} m(s,t) = \infty$ with positive probability, so $\nu(\mathbb{H}^2 \times A) = \infty$, which contradicts Theorem 3.4.

The proof now continues with the construction of a forest F, that is a graph without loops or cycles. Denote the set of encounter points by T, which is a.s. infinite by the above. We let each $t \in T$ represent a vertex $v(t)$ in F. For a given $t \in T$, let $U(t)$ be the unbounded component of C containing t. Then let k be the number of unbounded components of $U(t) \backslash S(t, R+1)$ and denote these unbounded components by C_1, C_2,..., C_k. For $i = 1$, 2, ..., k put an edge between $v(t)$ and the vertex corresponding to the encounter point in C_i which is closest to t in C (this encounter point is unique by the nature of the Poisson process).

Next, we verify that F constructed as above is indeed a forest. If v is a vertex in F, denote by $t(v)$ the encounter point corresponding to it. Suppose v_0, v_1, ..., $v_n = v_0$ is a cycle of length ≥ 3, and that $d_C(t(v_0), t(v_1)) < d_C(t(v_1), t(v_2))$. Then by the construction of F it follows that $d_C(t(v_1), t(v_2)) < d_C(t(v_2), t(v_3)) < \ldots < d_C(t(v_{n-1}), t(v_0)) < d_C(t(v_0), t(v_1))$ which is impossible. Thus we must have that $d_C(t(v_i), t(v_{i+1}))$ is the same for all $i \in \{0, 1, .., n-1\}$. The assumption $d_C(t(v_0), t(v_1)) > d_C(t(v_1), t(v_2))$ obviously leads to the same conclusion. But if $y \in Y$, the probability that there are two other points in Y on the same distance in C to y is 0. Hence, cycles exist with probability 0, and therefore F is almost surely a forest.

Now define a bond percolation $F_\epsilon \subset F$: Define C_ϵ in the same way as above. Let each edge in F be in F_ϵ if and only if both encounter points corresponding to its end-vertices are in the same component of C_ϵ. Since C_ϵ contains only bounded components, F_ϵ contains only finite connected components.

For any vertex v in F we let $K(v)$ denote the connected component of v in F_ϵ and let $\partial_F K(v)$ denote the inner vertex boundary of $K(v)$ in F. Since the degree of each vertex in F is at least 3, and F is a forest, it follows that at least half of the vertices in $K(v)$ are also in $\partial_F K(v)$. Thus we conclude

$$\mathbf{P}[x \in T,\, v(x) \in \partial_F K(v(x)) | x \in Y] \geq \frac{1}{2}\mathbf{P}[x \in T | x \in Y].$$

The right-hand side of the above is positive and independent of ϵ. But the left-hand side tends to 0 as ϵ tends to 0, since when ϵ is small, it is unlikely that an edge in F is not in F_ϵ. This is a contradiction.□

By Proposition 4.15, if $N_C = 0$, then $N_V = 1$ *a.s.* Thus we have an immediate corollary to Theorem 4.16.

Corollary 4.17 *At λ_c, $N_V = 1$ a.s.*

Next, we show the corresponding results for λ_u. Obviously, the nature of V is quite different from that of C, but still the proof of Theorem 4.18 below differs only in details to that of Theorem 4.16. We include it for the convenience of the reader.

Theorem 4.18 *At λ_u, $N_V = 0$ a.s.*

Proof. Suppose $N_V = 1$ a.s. at λ_u and denote the unbounded component of V by U. Then C contains only finite components *a.s.* by Proposition 4.15. Let $\epsilon > 0$ and let Z be a Poisson process independent of X with intensity ϵ. Let $C_\epsilon := \cup_{x \in X \cup Z} S(x, 1)$ and $V_\epsilon := C_\epsilon^c$. Since $X \cup Z$ is a Poisson process with intensity $\lambda_u + \epsilon$ it follows that C_ϵ has a unique unbounded component a.s. and hence V_ϵ contains only bounded components a.s. Let $\mathcal{V}_\epsilon$ be the collection of all components of V_ϵ. Define H_ϵ in the following way: For each $z \in \mathbb{H}^2$ we let $U_\epsilon(z)$ be the union of the components of $U \cap V_\epsilon$ being closest to z. We let each $h \in \mathcal{C} \cup \mathcal{V}_\epsilon$ be in H_ϵ if and only if $\sup_{z \in h} d(z, U) < 1/\epsilon$ and $U_\epsilon(x) = U_\epsilon(y)$ for all $x,\, y \in h$. Then,

$$\lim_{\epsilon \to 0} \mathbf{P}[0 \in H_\epsilon] = 1.$$

As in the proof of Theorem 4.16 this is enough to conclude that for ϵ small enough, H_ϵ contains an unbounded component with positive probability, and therefore V_ϵ contains an unbounded component with positive probability, a contradiction.

Now suppose that $\lambda = \lambda_u$ and $N_V = \infty$. Then also $N_C = \infty$ by Proposition 4.15. Therefore, for $x \in \mathbb{H}^2$, we can choose $R > 1$ such that the intersection of the two independent events

$$A(x, R) :=$$

$$\{S(x, R) \text{ intersects at least 3 disjoint unbounded components of } C[S(x, R)^c]\}$$

and $B(x, R) := \{|X(S(x, R))| = 0\}$ has positive probability. Next, suppose that Y is a Poisson process independent of X with some positive intensity. We call $y \in \mathbb{H}^2$ an *rendezvous point* if

- $y \in Y$;
- $A(y, R) \cap B(y, R)$ ocurrs;

- $S(y, 2R) \cap Y = \{y\}$.

By the above discussion,

$$\mathbf{P}[y \text{ is an rendezvous point } | \, y \in Y] > 0.$$

If y is a rendezvous point, y is contained in an unbounded component U of V and $U \backslash S(y, R)$ contains at least 3 disjoint unbounded components. In the same way as in the proof of Theorem 4.16 an infinite forest F in which every vertex has degree at least three is constructed, the only difference being that the vertices in this case correspond to the rendezvous points.

Again we define a bond percolation $F_\epsilon \subset F$. Let V_ϵ be defined as above. Each edge of F is declared to be in F_ϵ if and only if both its end-vertices are in the same component of V_ϵ. Then F_ϵ contains only finite connected components a.s. Now with the same notion as in the proof of Theorem 4.16,

$$\mathbf{P}[y \in T, \, v(y) \in \partial_F K(v(y)) | y \in Y] \geq \frac{1}{2} \mathbf{P}[y \in T | y \in Y].$$

Letting $\epsilon \to 0$ leads to the desired contradiction. □

Again, Proposition 4.15 immediately implies the following corollary:

Corollary 4.19 *At λ_u, $N_C = 1$ a.s.*

4.4 Proof of Theorem 4.2

Here we combine the results from the previous sections to prove our main theorem.

Proof of Theorem 4.2: If $\lambda < \lambda_u$ then Proposition 4.15 implies $N_V > 0$ a.s. giving $\lambda \leq \lambda_c^*$. If $\lambda > \lambda_u$ the same proposition gives $N_V = 0$ a.s. giving $\lambda \geq \lambda_c^*$. Thus

$$\lambda_u = \lambda_c^*. \tag{4.10}$$

By Theorem 4.16 $N_C = 0$ a.s. at λ_c, so $N_V > 0$ a.s. at λ_c by Proposition 4.15. Thus by Theorem 4.18

$$\lambda_c < \lambda_c^*. \tag{4.11}$$

Hence the desired conclusion follows by (4.10), (4.11) and Lemma 4.3. □

Obviously, we can also consider the Poisson-Boolean model in $\mathbb{H}^2$ with any fixed radius R. However, the proof given here of Theorem 4.2 does not work for all R. Consider the proof of Lemma 4.8. That proof gives that if $H \in \mathfrak{H}$ and H only contains bounded components *a.s*, then

$$\mathbf{P}[0 \in H] \leq 2 \exp(2\lambda\pi - 2\lambda\pi \cosh R) \lambda\pi \sinh R.$$

Some calculus gives that this is bounded by 1 for all λ only if

$$R \geq \cosh^{-1}(1 - 2/(1 - \mathrm{e}^2)) \approx 0.772.$$

Thus if we consider a model with $R < \cosh^{-1}(1 - 2/(1 - e^2))$ there are intensities, for which Lemma 4.8 does not give anything.

However, given R, we strongly believe it is possible to show that if $\mathbf{P}[S(0, \tilde{R}) \subset H]$ is close enough to 1 for some suitable $\tilde{R}$, which would be the case in our applications, then H contains unbounded components with positive probability. The proof would be more involved than the proof of Lemma 4.8, since, for example, the probability that $S(0, \tilde{R})$ intersects more than one component of H is not negligable.

In $\mathbb{R}^d$ there is a scaling argument for the Poisson-Boolean model with fixed radius that makes it unnecessary to consider radii other than 1, see Proposition 2.10 in [15]. This is not the case in $\mathbb{H}^2$.

References

[1] K.S. Alexander, *The RSW theorem for continuum percolation and the CLT for Euclidean minimal spanning trees*, Ann. Appl. Probab. **6** (1996), no. 2, 466-494.

[2] K.B. Athreya and P.E. Ney, *Branching Processes*, Springer Verlag, 1972.

[3] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Critical percolation on any nonamenable group has no infinite clusters*, Ann. Probab. **27** (1999), 1347-1356.

[4] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Group-invariant percolation on graphs*, Geom. Funct. Anal. **9** (1999), 29-66.

[5] I. Benjamini and O. Schramm, *Percolation in the hyperbolic plane*, J. Amer. Math. Soc. **14** (2001), 487-507.

[6] I. Benjamini and O. Schramm, *Percolation beyond $\mathbb{Z}^d$, many questions and a few answers*, Electronic Commun. Probab. **1** (1996), Paper no. 8, 71-82 (electronic).

[7] R. M. Burton and M. Keane, *Density and uniqueness in percolation*, Comm. Math. Phys. **121** (1989), 501-505.

[8] J.W. Cannon, W.J. Floyd, R. Kenyon and W.R. Parry. Hyperbolic geometry. In *Flavors of geometry*, pages 59-115. Cambridge University Press, 1997.

[9] R. Durrett, *Probability: Theory and Examples (2nd ed.)*, Duxbury Press, 1995.

[10] G. Grimmett, *Percolation (2nd ed.)*, Springer-Verlag Berlin, 1999.

[11] O. Häggström, *Infinite clusters in dependent automorphism invariant percolation on trees*, Ann. Probab. **25** (1997), 1423-1436.

[12] O. Häggström and J. Jonasson, *Uniqueness and non-uniqueness in percolation theory*, 2005.

[13] P. Hall, *On continuum percolation*, Ann. Probab. **13** (1985), 1250-1266.

[14] J. Jonasson, *Hard-sphere percolation: Some positive answers in the hyperbolic plane and on the integer lattice*, 2001.

[15] R. Meester and R. Roy, *Continuum Percolation*, Cambridge University Press, New York, 1996.

[16] R.H. Schonmann, *Stability of infinite clusters in supercritical percolation*, Probab. Th. Rel. Fields **113** (1999), 287-300.